Math 150, *Lecture Notes- Bonds* Name____________

Section 3.4 *Concavity and the Second Derivative Test*

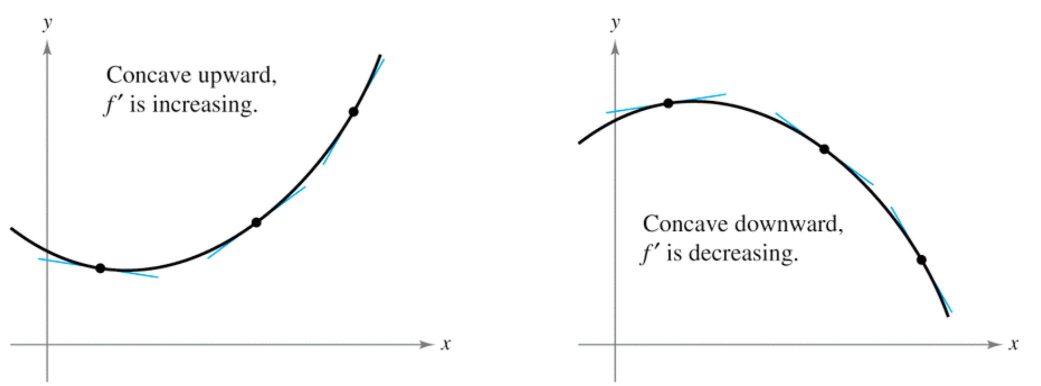
Concavity

You have already seen that locating the intervals in which a function f increases or decreases helps to describe its graph. In this section, you will see how locating the intervals in which f' increases or decreases can be used to determine where the graph of f is *curving upward* or *curving downward*.

Definition of Concavity

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.

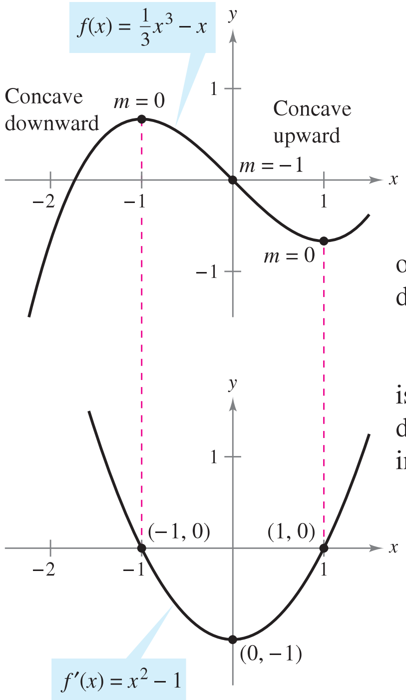
The following graphical interpretation of concavity is useful.



(a) The graph of f lies above its tangent lines. **(b)** The graph of f lies below its tangent lines.

Figure 3.24

1. Let f be differentiable on an open interval I. If the graph of f is concave *upward* on I, then the graph of f lies *above* all of its tangent lines on I.
   [See Figure 3.24(a).]
2. Let f be differentiable on an open interval I. If the graph of f is concave *downward* on I, then the graph of f lies *below* all of its tangent lines on I.
   [See Figure 3.24(b).]



f' is decreasing. f' is increasing.

The concavity of f is related to the slope of the derivative.

Figure 3.25

To find the open intervals on which the graph of a function f is concave upward or concave downward, you need to find the intervals on which f' is increasing or decreasing. For instance, the graph of

$$f(x) = \tfrac{1}{3}x^3 - x$$

is concave downward on the open interval $(-\infty, 0)$ because $f'(x) = x^2 - 1$ is decreasing there. (See Figure 3.25.) Similarly, the graph of f is concave upward on the interval $(0, \infty)$ because f' is increasing on $(0, \infty)$.

THEOREM 3.7 Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward in I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward in I.

To apply Theorem 3.7, locate the x-values at which $f''(x) = 0$ or f'' does not exist. Second, use these x-values to determine test intervals. Finally, test the sign of $f''(x)$ in each of the test intervals.

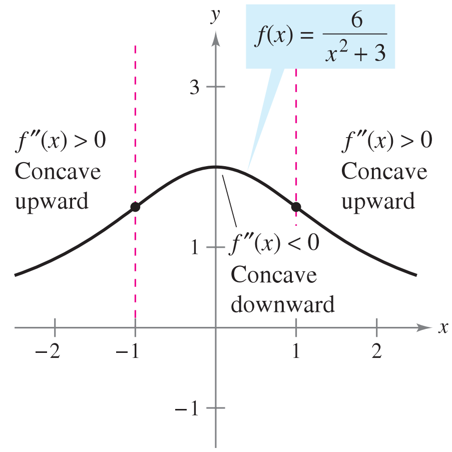
NOTE A third case of Theorem 3.7 could be that if $f''(x) = 0$ for all x in I, then f is linear. Note, however, that concavity is not defined for a line. In other words, a straight line is neither concave upward nor concave downward.

Ex.1

Determine the open intervals on which the graph of

$$f(x) = \frac{6}{x^2 + 3}$$

is concave upward or downward.



From the sign of f'' you can determine the concavity of the graph of f.
Figure 3.26

Interval			
Test Value			
Sign of $f''(x)$			
Conclusion			

The function given in Example 1 is continuous on the entire real line. If there are x-values at which the function is not continuous, these values should be used, along with the points at which $f''(x) = 0$ or $f''(x)$ does not exist, to form the test intervals.

Ex.2

Determine the open intervals on which the graph of $f(x) = \frac{x^2 + 1}{x^2 - 4}$ is concave upward or concave downward.

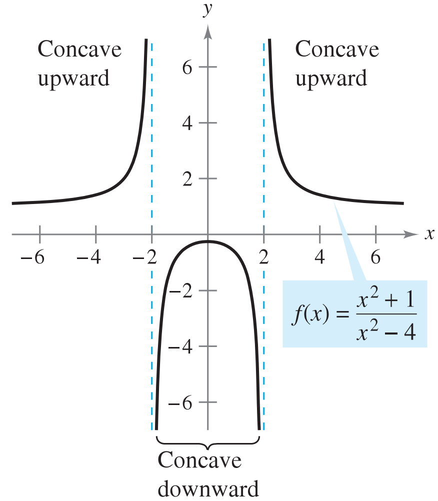
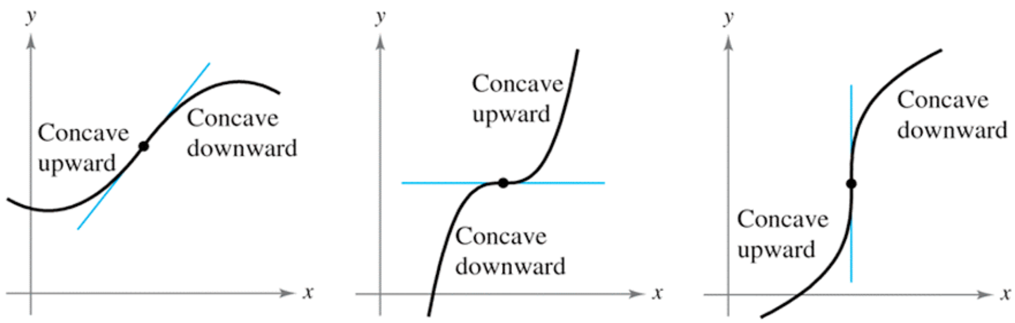


Figure 3.27

Interval			
Test Value			
Sign of $f''(x)$			
Conclusion			

Points of Inflection

The graph in Figure 3.26 has two points at which the concavity changes. If the tangent line to the graph exists at such a point, that point is a **point of inflection.** Three types of points of inflection are shown in Figure 3.28.



The concavity of f changes at a point of inflection. Note that a graph crosses its tangent line at a point of inflection.

Figure 3.28

> **Definition of Point of Inflection**
>
> Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

NOTE The definition of *point of inflection* given above requires that the tangent line exists at the point of inflection. Some books do not require this. For instance, we do not consider the function

$$f(x) = \begin{cases} x^3, & x < 0 \\ x^2 + 2x, & x \geq 0 \end{cases}$$

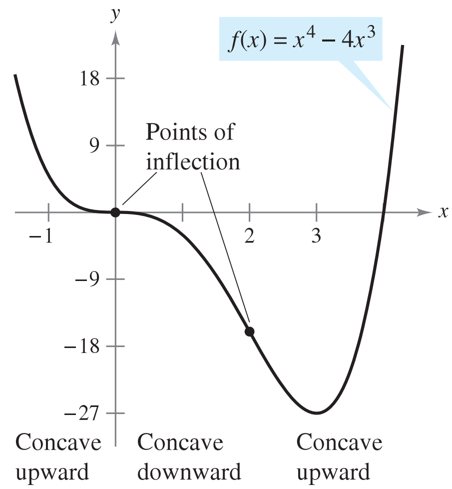
to have a point of inflection at the origin, even though the concavity of the graph changes from concave downward to concave upward. ■

To locate *possible* points of inflection, you can determine the values of x for which $f''(x) = 0$ or $f''(x)$ does not exist. This is similar to the procedure for locating relative extrema of f.

> **THEOREM 3.8 Points of Inflection**
>
> If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' does not exist at $x = c$.

Ex.3 Determine the points of inflection and discuss the concavity of the graph of

$f(x) = x^4 - 4x^3$.

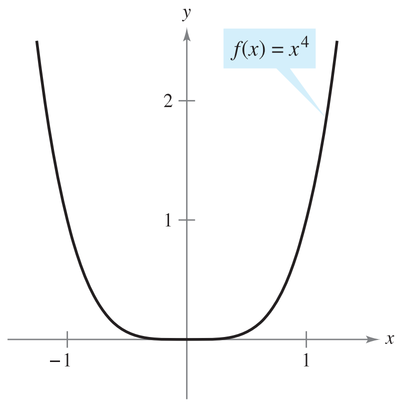


Points of inflection can occur where $f''(x) = 0$ or f'' does not exist.

Figure 3.29

Interval			
Test Value			
Sign of $f''(x)$			
Conclusion			

The converse of Theorem 3.8 is not generally true. That is, it is possible for the second derivative to be 0 at a point that is *not* a point of inflection. For instance, the graph of $f(x) = x^4$ is shown in Figure 3.30. The second derivative is 0 when $x = 0$, but the point $(0, 0)$ is not a point of inflection because the graph of f is concave upward in both intervals $-\infty < x < 0$ and $0 < x < \infty$.



$f''(x) = 0$, but $(0, 0)$ is not a point of inflection.

Figure 3.30

The Second Derivative Test

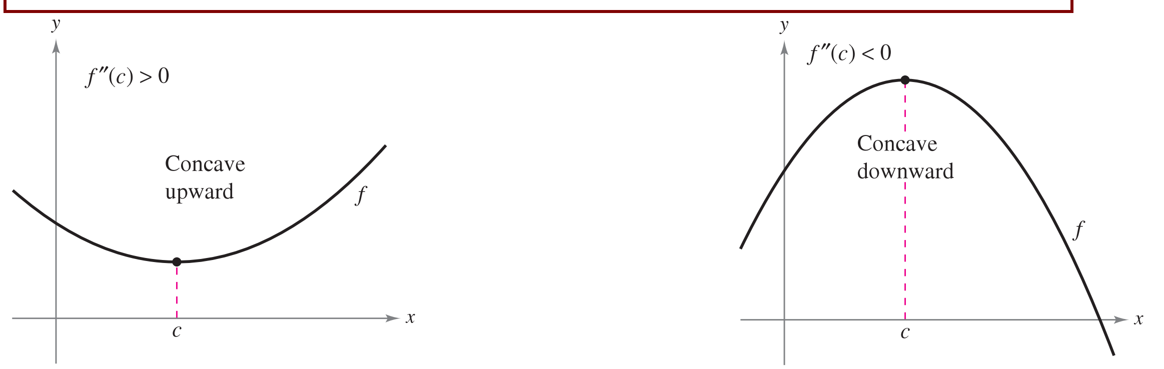
In addition to testing for concavity, the second derivative can be used to perform a simple test for relative maxima and minima. The test is based on the fact that if the graph of a function f is concave upward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be a relative minimum of f. Similarly, if the graph of a function f is concave downward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be a relative maximum of f (see Figure 3.31).

THEOREM 3.9 Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum at c, a relative minimum at $(c, f(c))$, or neither. In such cases, you can use the First Derivative

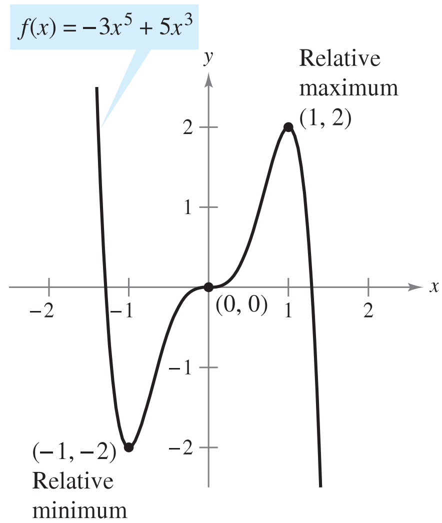


If $f'(c) = 0$ and $f''(c) > 0$, $f(c)$ is a relative minimum.

Figure 3.31

If $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a relative maximum.

Ex.4 Find the relative extrema for $f(x) = -3x^5 + 5x^3$.



(0, 0) is neither a relative minimum nor a relative maximum.

Figure 3.32

Point			
Sign of $f''(x)$			
Conclusion			

Because the Second Derivative Test fails at $(0, 0)$, you can use the First Derivative Test and observe that f increases to the left and right of $x = 0$. So, $(0, 0)$ is neither a relative minimum nor a relative maximum (even though the graph has a horizontal tangent line at this point). The graph of f is shown in Figure 3.32. ■

Ex.5 Use the Second Derivative Test, where applicable, to find the relative extrema of $h(\theta) = 2\sin(\theta) + \cos(2\theta)$ on the interval $[0, 2\pi]$.

Point				
Sign of $f''(x)$				
Conclusion				